Algunas Aplicaciones del Método de las Soluciones Fundamentales para Modelar el Flujo de Fluidos en Medios Porosos

Some Applications of the Fundamental Solution Method for Modelling Fluid Flow in Porous Media

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RESUMEN

El método de las soluciones fundamentales es una técnica numérica libre de mallas para resolver ecuaciones diferenciales parciales. Su implementación combina la función libre de Green, una generalización del método de las imágenes, la descomposición en valores singulares y una adaptación del método de colocación a lo largo de los bordes, lo que produce un algoritmo sencillo para la simulación de algunos problemas en ciencias e ingeniería. Se presenta una revisión de su adaptación a la solución de tres problemas clásicos de mecánica de fluidos en medios porosos. Estos problemas son: el flujo de trazadores, la distribución de presión no estática en un yacimiento y la determinación de líneas equipotenciales o de flujo en yacimientos con barreras geológicas muy finas. Los resultados evidencian que el método de las soluciones fundamentales es realmente una nueva alternativa entre los métodos numéricos para simular problemas simples en yacimientos y flujo de agua subterránea.

ABSTRACT

The fundamental solution method is a meshless approach for the numerical solution of partial differential equations. Its implementation combines free Green functions, a generalization of image's method, the singular value decomposition, and a boundary collocations method to produce a very simple algorithm for the simulation of some problems in science and engineering. A review of its adaptation to the solution of three classical problems for fluid flow in porous media is presented. These problems are: the tracer flow in a reservoir, the transient pressure distribution in an aquifer, and the computation of pressure and streamlines in a reservoir with thin geological barriers. The analysis of the results shows that the fundamental solution method is a new numerical alternative for the simulation of simple reservoirs or groundwater flow problems.

Palabras clave (keywords): soluciones fundamentales, descomposición en valores singulares, pozos imaginarios, trazador, presión, líneas de flujo, barreras geológicas. (fundamental solutions, singular value decomposition, imaginary wells, tracer, pressure, stream-lines, geological barriers).

1 INTRODUCTION

The method of fundamental solutions (MFS) is a special kind of boundary integral method for the solution of partial differential equations. It was originally developed in the Soviet Union [1] in the early sixties and its first numerical implementation was reported in [2]. Since that time many versions of error estimates and convergence analysis have been presented in the literature, see for example [3, 4, 5]. All those analysis and more updated versions, see [7], are restricted to particular cases of equations and geometries. In fact, a general framework for an abstract convergence analysis of the MFS is a topic of current research at this time. On the other hand, numerical implementations of the MFS have been applied to a variety of physical problems such as: acoustic, elasticity, electormagnetism, and heat transfer. Our applications belong to the general area of fluid dynamics. A full review of these applications and their analysis can be found in the survey article [6] and in [7], which contains a complete collection of the most recent results for the MFS. However, all these references do not report specific applications of the MFS to fluid flow problems in porous media or in the reservoir engineering context. Therefore, this article fills that gap and describes three particular application of the MFS to reservoir engineering problems. They are the simulation of tracer flow in a reservoir, the computation of transient pressure distribution in a reservoir with a compressible fluid, and the calculation of streamlines distributions in a reservoir with very thin barriers. Two of these application will be briefly reviewed. They have been fully developed in [8, 9], but they are not cited in the general surveys [6, 7] . The third application is new and it provides original results, which complement those reported in [10].

This paper has been divided in six sections. The first section is this introduction. Second section contains a general description of the MFS. Third, fourth and fifith sections are devoted to the applications. Finally, the conclusions and discussions are presented in section six.

2 METHOD OF FUNDAMENTAL SOLUTION

In its most general form the MFS is a numerical technique for solving linear boundary value problems of the form

$$Lu(p) = Sources \tag{1}$$

with boundary conditions

$$Bu(p) = g \tag{2}$$

In these equations the *sources* and g are given functions, u is the unknow function with parameter p as its independent variable, L is linear differencial operator, and B is a linear boundary condition. Notice that most linear evolution equations can be solved as a sequence of linear boundary value problems in time, so our presentation of the MFS will be restricted to equations (1) and (2). By a fundamental solution of equation (1), we mean a function $K(p,q)$ such that

$$LK(p,q) = \delta(p,q) \tag{3}$$

where δ(*p,q*) is the Dirac delta function. This function is defined everywhere except when $p = q$, where it is singular. Our version of the MFS assumes that an approximated solution u_N to equation (1) can be represented by the following relation.

$$u_N(p) \equiv \sum_{k=1}^{N} Q_k K(p,q_k) + PS(p) \tag{4}$$

The summation term in the above expression represents a finite linear combination of fundamental solutions with singularities at the points q_k. These singularities are called imaginary wells in our implementation, in analogy to the method of images, and they are located outside of the reservoir or domain of integration denoted by Ω. Figure 1 shows these singularities as black circles. The term *PS*(*p*) is a particular solution of equation (1), which can be determined by simple simple analytic expressions for the three applications that will be study in the next sections. For example, in the specific case of Figure 1 the particular solution will be the sum of the fundamental solutions associated to the four real wells, denoted with circle with arrows in Figure 1, whose rates are known. Since MFS's approximation (4) satisfies equation (1) for all values of the the imaginary wells rates, $\{Q_k\}$, which are unknown, then these rates are sougth by enforcing the boundary condition (2). This is achieved by minimizing the following energy integral.

$$I(Q_1,\cdots,Q_N) \equiv \int_{\partial\Omega} (Bu_N(p) - g(p))^2 dp \tag{5}$$

In all MFS's implementations this integral is approximated by some quadrature rule, which generates a set of collocation points points, represented by black squares in Figure 1, along the reservoir boundary. It can be shown that minimization of the energy integral approximation produces a linear system

$$A \cdot Q = b \tag{6}$$

where $Q = (Q_1, ..., Q_N)^t$ is a column vector with the imaginary wells rates, b is a column vector whose entries are known, and matrix A has dimension MxN with M the number of collocation points.

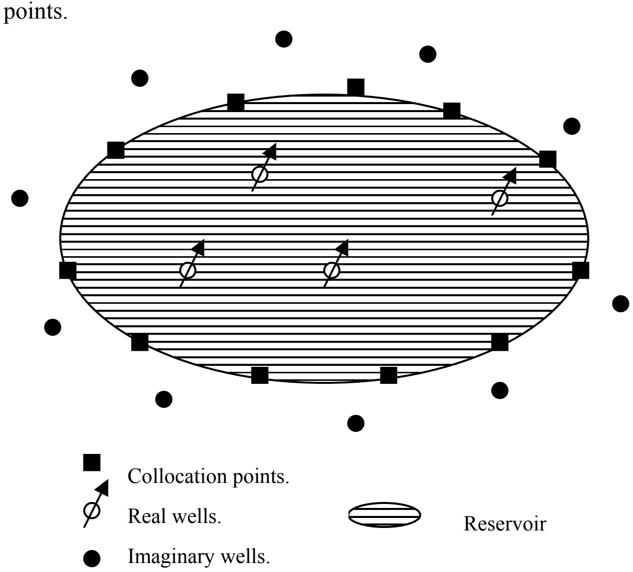


Figure 1: MFS's configuration in a generic reservoir.

In general, system (6) is overdetermined because the number of collocations points M is always greater o equal than the number of imaginary wells N. It is well known that matrix A condition number grows very fast as N is increased. Therefore, solution of system (6) is viewed as a least square problem and QR factorization or singular value decomposition (SVD) are the usual choices for solving (6). It is known that QR is faster than SVD, because it uses QR factorization in its own algorithm. However, the SVD is a more robust solver for dealing with the linear systems generated by the MFS [11]. The SVD of A produces a factorization USV where $U= [U_1,...,U_M]$ and $V=[V_1,...,V_N]$ are unitary matrices of order MxM and NxN, and S is a diagonal matrix of order MxN with the singular values σ_i along its diagonal entries. The final solution of (6) as a function of the SVD of A can be represented as

$$Q = \sum_i ((b \cdot U_i)/\sigma_i) V_i \tag{7}$$

where the number of terms in the above summation is a function of the spectral cutoff .

3 FIRST APPLICATION (TRACER FLOW)

The mathematical model for tracer flow in a reservoir is a system of two partial differential equations. One of them is a diffusion convection equation for the tracer concentration, while the second equation of elliptic type is named pressure equation which models the pressure behavoir in the reservoir. In [8] a novel method for solving this system was proposed. It uses the MFS for solving the pressure equation and a numerical version of the method of characterictics is used for solving the tracer concentration. This method is very general and efficient. However, our application in this section will be restricted to an irregular homogeneous reservoir with an incompressible fluid, which allows us to show the easy implementation of the MSF in problems with irregular regions. This feature is usually overlooked and it is one of the main advantages of the MSF. The synthetic reservoir is displayed in Figure 2(a). It has a peanut sheaped boundary and five real wells in its interior. Four of these wells are injectors (I) and fifth well is a producer (P) at the reservoir center. Under the assumption of fluids incompressibility, well's rates must be in balance. For this example injections rates are a quarter of the production rate. In this case the pressure equation in (1) takes the following form.

$$(kh/\mu)\Delta u(p) = q_P \delta(p,p_P) - \sum_{I=1}^{4} q_I \delta(p,p_I) \tag{8}$$

This is a Poisson equation where the cosntant kh/μ is the ratio between the reservoir permeability times its thickness and fluid viscosity, q*'s* are the real wells rates, and u is the pressure.

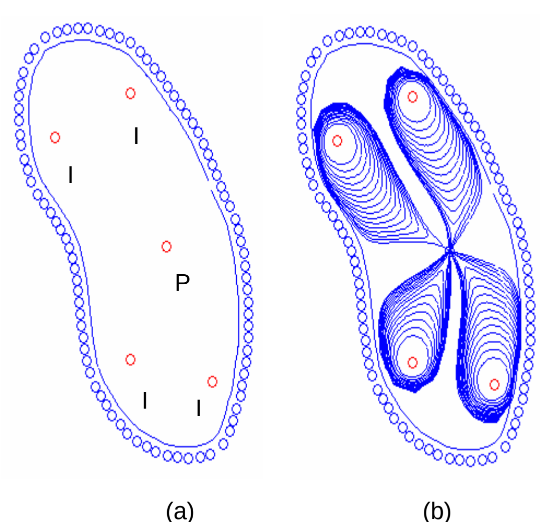


Figure 2: (a) Reservoir geometry and imaginary wells around its boundary. (b) Tracer front evolution computed with the MFS.

The MFS approximations (4) associated to (8) has a particular solution given below.

$$PS(p) = \frac{\mu q_p}{2\pi kh}\ln|p - p_P| - \sum_{I=1}^{4}\frac{\mu q_I}{2\pi kh}\ln|p - p_I| \quad (9)$$

The reservoir boundary is impermeable, which is represented by a homogeneous Neumann condition. This means that operator B is the directional derivative of u in the direction of exterior normal to the reservoir boundary. With this information the MFS was applied to (8) when the number of imaginary wells was between seventy and one hundred. In this case the fundamental solution is

$$K(p,q) = \frac{\mu}{2\pi k}\ln|p - q|. \quad (10)$$

Imaginary wells distance to the reservoir boundary was chosen by trial and error. In general, that distance should be always less than the real wells distance to the boundary and it must be uniformly distributed. Figure 2 (a) shows one of the configurations used to generate the approximated pressure u_N. The number of collocation points was approximately one hundred and twenty. A linear system (6) with dimension 120x(70 to 100) was generated and solved with SVD. Darcy's velocity was computed analytically from u_N and it was used to obtain the streamline distribution in the reservoir. Since pressure and concentration equations are loosely coupled then Darcy's velocity and streamline distribution need to be computed only one time at the beginning of the simulations. It is proved that concentration equation restricted to each streamline reduces to a one dimensional convection-diffusion equation. Futhermore, the time of flight along each streamline allows us to reduce the concentration equation to a convection equation with constant velocity. Moreover, this application and our work [8] assume a null diffusion coefficient. In that case the concentration equation becomes strictly hyperbolic and tracer front evolution in time can be easily obtained by the method of the characteristic as showed in Figure 2(b). Tracer solutions displayed in Figure 2(b) is of great quality, the MFS combined with the method of characteristic produces a semianalytic solution that can not be improved by more general numerical methods such as finite differences or finite elements. It is worth to mention that our assumption of null diffusion term transform our application in a very difficult test for standard numerical schemes. In the cases of no null diffusion coefficient our scheme still works, but it is more difficult to visualize the tracer front because it is not longer a sharp discontinuity. The application presented in this section was specially designed for this article. It is simplify version of a more general case studied in [8], where a detailed analysis on more general conditions and validations tests can be found. Our main objective was to give evidence that the MFS is able to simulate a tracer flow problem in a reservoir with a very simple algorithm.

4 SECOND APPLICATION (TRANSIENT PRESSURE)

In our previous application reservoir resident fluid and tracer were assumed incompressible. In general, such supposition is not valid. For example, fluids in a petroleum reservoirs should be simulated as slightly compressible in the early stages of oil production. If the reservoir fluid is compressible then its pressure equations is a diffusion or diffusivity equation. In this application the MFS will be applied to an irregular shaped, two dimensional, homogeneus reservoir with constant fluid and geological properties. The reservoir compressibility will be in the order of 10^{-5} 1/psi . It is assumed that eight production wells have been perforated simultaneausly at random locations in the reservoir. Theses wells and reservoir geometry are represented in Figure 3 (a). Morevore, it is assumed that the eight productions wells extract fluid at constant and equals rates all . Under this conditions equations (1) takes the following expression.

$$\left(\frac{\phi\mu c}{K}\frac{\partial u}{\partial t} - \Delta u\right)(p) = \sum_{n=1}^{8}\frac{\mu q_n}{Kh}\delta(p - p_n) \quad (11)$$

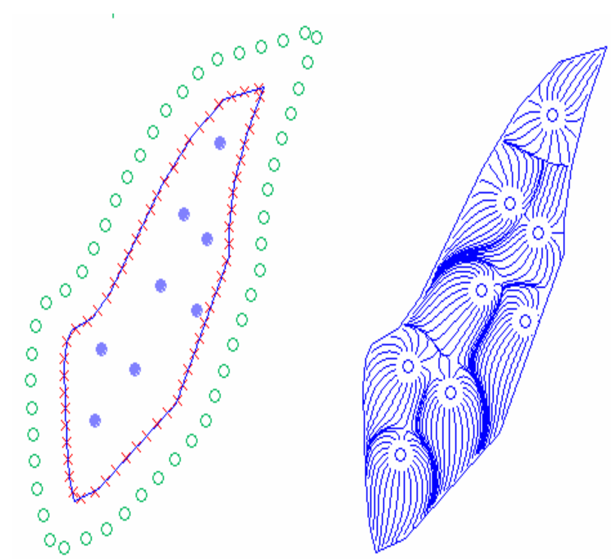
In this equation u is the reservoir pressure, ϕ is porosity, μ is fluid viscosity, c is fluid compressibility, h is reservoir thickness, K is reservoir permeability, q_n's are the well's rates, and p_n's are the real wells positions. For additional information about geological and fluid properties we refer to our previous publication [9]. At the reservoir boundary a no flow boundary condition is imposed, which gives a null Neumann condition for the operator B in (2). Since equation (11) is time dependent then its fundamental solutions has a more complex form

$$K(p,q,t) = \frac{\mu}{4\pi Kh} Ei\left(\frac{c\phi\mu|p - q|^2}{4Kt}\right). \quad (12)$$

where $Ei(x)$ is the exponential integral function defined by $\int_x^\infty (e^{-y}/y)dy$ and t is the time. This expression allows us to write a particular solution for (11)

$$PS(p,t) = u_0 + \sum_{n=1}^{8}\frac{\mu q_n}{4\pi Kh} Ei\left(\frac{c\phi\mu|p - p_n|^2}{4Kt}\right). \quad (13)$$

in which u_0 is the initial reservoir pressure. Equations (12) and (13) are used in (4) to obtain the MFS approximation of pressure as a function of time $u_N(p,t)$. For that approximations eighty imaginary wells and one hundred collocation points were employed. A scketch of the imaginary wells and collocations positions in the reservoirs is presented in Figure 2. The pressure approximation was computed with the MFS up to the time when the reservoir has reached the steady state condition.



(a) (b)

Figure 3: (a) Reservoir configuration: real wells (circles inside the reservoir), imaginary wells (circles outside the reservoir), collocation points (crosses along the boundary). (b) Streamline distirbution shows areas drained by each production well.

The steady state condition has been reached in a reservoir if the rate of change in reservoir pressure as a function of time is constant. Under steady state condition the drained areas of each wells can be determined from the well's pressure derivate in time. In our example such derivate is identical on each well. This was expected because all wells are producing at the same rate. Qualitatevely this conclusion can be observed by computing the streamlines distribution in the reservoir when it has reached the steady state condition. In our application, this distribution was computed from the MFS pressure approximations and it is represented in Figure 2 (b). Since all wells are producers then the streamlines associated to each well convers all the area drained by it. Figure 2 (b) shows that areas darined by each well are equal, which gives evidence that pressure approximation obtained from the MFS is correct. More information about this application of the MFS can be found in [9].

5 THIRD APPLICATION (FLOW WITH THIN BARRIERS)

Mathematical models for the simulation of thin barriers is a standard topic in reservoir engineering. It has been recognized that general numerical methods based on domain discretizations are inefficients in dealing with this type of problems. One of the main reason for this deficiency is that domain methods need very refined grids for the thin barriers. Our application in this sections provides an adaptation of the MFS for computing streamline distribution in reservoirs with thin barriers. A key idea in our approach is to recognize that barriers can be represented as singularities of the reservoir pressure. This idea allows us to eliminate the problem of scales that leads to the use of very refined grids in domain methods. In this application is assumed that barriers and wells are the only source in the reservoir pressure and outside them geological and fluid properties are constant. Moreover, the fluid will be incompressible. It is know that a mathematical representation of the complex potential associated to a horizantal barrier with its center at the origen is

$$BA(z,a,\alpha,V_0)\equiv -V_0(z\cos(\alpha)-i\sqrt{z^2-4a^2}\,\sin(\alpha)) \quad (14)$$

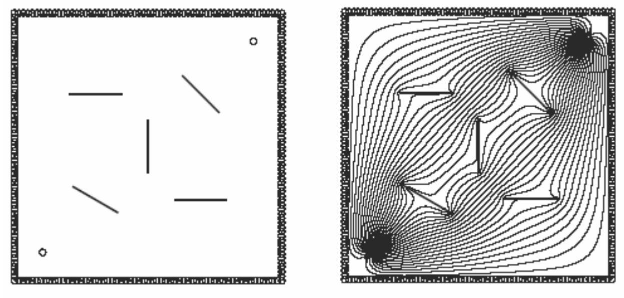
where α is the angle between the barrier and the velocity field V_0, $z = x + iy$ is a complex variable for position, and a is a positive number for defining half of the barrier length. In Figure 4 (a) shows the reservoir configuration that will be used in this application. It contains two wells, one injector and one producer with balanced rates, and five thin barriers with the same length but with random orientation. Under these assumptions the pressure equation, equation (1) in the MFS context, is a Poisson equation.

$$\Delta u(p)=\sum_{i=1}^{2} q_i\delta(p,p_i)+\Delta\left[\mathrm{Re}\left(\sum_{i=1}^{5}T_i(BA(p,a_i,\alpha_i,V_{oj})\right)\right] \quad (15)$$

In this relation Re means real part, the first summation term represents the real wells inside the reservoir, the second summation term comes from the five source generated by the barriers, and T_i are appropiated Mobius transformations associated to each barrier position. At the reservoir edge a no flow condition is imposed to obtain the same homegeneous Neumann condition for operator B in (2). The MFS approximation for the pressure u_N has a particular solution.

$$PS(p)=\sum_{i=1}^{2}\frac{q_i}{2\pi}\ln|p-p_i|+\mathrm{Re}\sum_{i=1}^{5}T_i(BA(p,a_i,\alpha_i,V_{oi})) \quad (16)$$

One hundred and twenty imaginary wells and one hundred and forty collocation points were used in (4) to obtain a MFS pressure approximation. This pressure was used to compute the streamlines distribution displayed in Figura 4 (b). These streamlines satisfy the no flow boundary condition along all the boundaries. Around the injections and production wells the streamline distribution is prefect, but very few of them are covering the area between wells and reservoir corners. This lack of resolution can be solved by increasing the number of streamlines, but it is not showed the Figure.



(a) (b)

Figura 4: (a) Square shaped reservoir with five random barriers and two real wells. Imaginary are very close to the boundary. (b) Streamline distribution computed with the MFS.

An analytic solution for the reservoir configuration used in this application is not available. In order to validate the MFS solution then the same reservoir problem was solved with standard finite difference method on a very refined grid (200x200 cells). This is the main reason for using a square shaped reservoir. The finite difference solution was used as reference or analytic solution to evaluate the precision of the MFS approximations. A comprative study of convergence of the MFS is presented in the Table 1.

Table 1: Convergence Study

No. imaginary wells	No. collocation points.	Condition A	Error (max-norm)
20	40	45.60	1.4400
52	72	246.29	0.7100
100	120	$2.08\text{x}10^3$	0.0500
172	192	$1.93\text{x}10^5$	0.0002

First column gives the number of imaginary wells, second column provides the number of collocation points used by the MFS, third column shows the condition number of matrix A in system (6) and the last column display the maximum error between the MFS approximation and the finite difference solution on the very refined grid. Several observation can be made from Table 1. The number of collocation points was always twenty points greater than the number of imaginary. It was found that greater number of collocations did not have strong effect on the errors. From the MFS's description is clear that the number of collocation points depends on the number of imaginary wells. On the other hand, a review of the first and fourth columns shows that increments in the number of imaginary wells reduce the error in maximum norm between the MFS approximation and the analytic solution. This means that the number of imaginary wells has strong effect on the MFS convergence. The third column shows a characteristic property of the MFS, increments in the number of imaginary wells produce huge increments in the condition number of matrix A. This obervation is one of the weakest point of the MFS, but it has been alleviated by the use of SVD. As long as the condition number of A does not exceed the machine precision then increments in the number of imaginary wells will increase the accuracy of the MFS approximation. In our application problem two hundred imaginaries wells will produce an MFS approximations which differs from the exact solution by a maximum error of 10^{-4} . Our analysis provides evidence that MFS approximations obtained in this application converge to the exact solution when the number of imaginary wells is increased. This result agrees with theoretical convergence analysis of the MFS. Therefore, we may conclude that the streamline distribution computed in this application by the MFS is correct and accurated.

6 CONCLUSIONS AND DISCUSSION

This article has described three applications of the MFS to specific problems in reservoir engineering or fluid flow in porous media. The great advantage of the MFS over other methods is the ease with which it can be implemented for problems on irregular domains and simple non homogeneous terms in the equations. These observations are also valid for three dimensional problems, although our research work has been limited to two dimensional cases. Sometimes these advantages are overlooked by people who actually compute. It is also clear that the MFS has still many limitations. In particular, the MFS for time dependent problems should be improved and it is certainly a research area. Similarly, an important research area is the development of efficient solvers for system generated by the MFS. On the other hand, our three applications give evidence that the MFS is sufficiently mature as a numerical technique to solve static or elliptic problems in porous media and reservoir engineering problems. In this context the MFS should be an alternative choice to the standard boundary element method. Its application in combination with more general numerical methods is a topic of current research in all areas of engineering.

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