The effect of density dependent emigration on spread of infectious diseases: a modelling study

El efecto de la densidad dependiente de la emigración en la propagación de enfermedades infecciosas: un estudio de modelos

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Abstract

In this study, we proposed and analyzed an SIS mathematical model by considering population density dependent emigration. It is assumed that the disease is transmitted by direct contact of infective and susceptible populations. We also assumed that the rate of contact is emigration dependent i.e. contact rate is variable which depends on the current population of habitat as well as on non-emigrating population density of habitat. The equilibria and their stability are studied by using the stability theory of differential equations and simulation. The model analysis shows that the spread of infectious disease in habitat decreases if the rate of emigration increases but it increases as the population density of non-emigrating population increases. The simulation study of the model confirms these analytical results.

Kew words: Modelling and Simulation, Mathematical model, Density dependent emigration, Stability.

Resumen

En este estudio, propusimos y analizamos un modelo matemático SIS considerando la densidad de población dependiente de la emigración. Se supone que la enfermedad se transmite por contacto directo de poblaciones infectivas y susceptibles. También asumimos que la tasa de contacto depende de la emigración, es decir, la tasa de contacto es variable y depende de la población actual del hábitat, así como de la densidad de la población del hábitat que no emigra. Los equilibrios y su estabilidad se estudian utilizando la teoría de la estabilidad de las ecuaciones diferenciales y la simulación. El análisis del modelo muestra que la propagación de enfermedades infecciosas en el hábitat disminuye si la tasa de emigración aumenta, pero aumenta a medida que aumenta la densidad de población de la población no emigrante. El estudio de simulación del modelo confirma estos resultados analíticos.

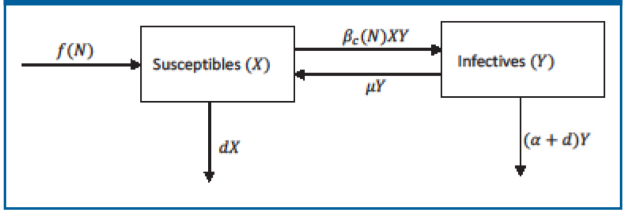
Palabras nuevas: modelado y simulación, modelo matemático, emigración dependiente de la densidad, estabilidad.

Introduction

The spread of infectious diseases in human population depend upon various factors such as the densities of susceptibles and infectives, their contact rate, environmental and ecological factors, etc. Many human infectious diseases in nature transmit by both direct and indirect physical contacts such as tuberculosis, influenza, conjunctivitis, AIDS, Hepatitis and typhoid fever. The spread of infectious diseases also depends on the emigration of population from the human habitat as well as its immigration to the habitat.

Mathematical models are important tools in the study of spread and control of infectious diseases. Many researchers have been considered the constant contact rates between susceptible and infective to study disease dynamics by considering various mathematical models[2-11,17]. But the population density dependent rate of contact plays a key role in the spread of infectious diseases[1,14]. The effect of variable death rate has also been considered[12,14]. It is noted that the effect of fraction of the population in the habitat which may emigrated or not has not been studied in most of the mathematical models related to the disease dynamics. It is further pointed out that the effect of density dependent emigration by considering the effect of non-emigrating population of the habitat has also not been studied.

Fig 1. Schematic flow of the Disease



In this paper, therefore, the following effects are studied on the spread of infectious diseases by proposing a non-linear mathematical model.

(1) The effect of population density dependent emigration, which is considered as a linear function of non-emigrating population of the habitat.

(2) The effect of emigration dependent contact rate between susceptibles and infectives, which is a linear function of non-emigrating population of the habitat.

Let be the density of population in the habitat. The variable emigration function of human population density with constant immigration is proposed as follows.

$$f(N)=A-\mu_0 (N-N_0);\quad A>0,N_0\geq 0 \qquad (1)$$

In eq. (1), is the non-emigrating population density of habitat and is the coefficient rate of emigration from habitat. It is noted from eq. (1) that increases as increases but it decreases as increases.

Further, we also assume that the contact rate between susceptible and infectives is also emigration dependent. Thus, the emigration dependent contact rate is proposed as follows:

$$\beta_{-e}(N) = \beta - \beta_0 (N-N_0). \qquad (2)$$

Where, is the constant contact rate, is the coefficient rate of emigration. It is noted that as increases increases but it decreases as increases.

The basic objectives of this study to incorporate the effects of non-emigrating population as well as the density dependent contact rates in the modelling process and to study its effect on infectious disease dynamics.

SIS Model with density dependent emigration: Let N be the total human population density of the habitat, which consist the susceptible population density X and infective population density Y. In view of the above, the dynamics of model is governed by following system of non-linear differential equations:

$$dX/dt=A-\mu_0 (N-N_0)-(\beta-\beta_0 (N-N_0))XY-dX+\mu Y \qquad (3)$$

$$dY/dt=(\beta-\beta_0 (N-N_0))XY-(\mu+\alpha+d)Y$$

$$N(0)>0,X(0)>0 \text{ and } Y(0)\geq 0$$

In the model system (3), is natural death rate of human population, is death rate coefficient of infective human population due to disease related factors, is the recovery rate of infective human population density.

Equilibrium Analysis: To analyze this model system (3), we reduced into the equivalent form of model system (3) by taking and using

$$dY/dt=(\beta-\beta_0 (N-N_0))(N-Y)Y-(\mu+\alpha+d)Y \qquad (4)$$

$$dN/dt=A_0- d_0 N-\alpha Y$$

The following lemma is needed for further analysis of model system (4).

Lemma 3.1. The region of attraction of model system (4) is given by the set

$$\Omega=\{(Y,N) \in R^2 ;\ 0\leq Y\leq N_{min} \text{ and } N_{min}= =A_0/(\alpha+d_0)\leq N\leq A_0/d_0 =N_{-max}\}$$

Which attract all the solution of model system (4) in the positive quadrant of the region

Theorem 3.1. The model system (4) has the following two non-negative equilibria in .

(i)) $E_0 (0,\ A_0/d_0)$, The disease free equilibrium.

(ii) $E_1=E_1 (Y^*,N^*)$ is endemic equilibrium which exists if .

Here is the basic reproduction number.

Proof: The existence of disease free equilibrium point is obvious we prove the existence of from the model system (4). Let then and are given from the following equations.

$$(\beta-\beta_0 (N-N_0))(N-Y)-(\mu+\alpha+d)=0 \qquad (5)$$

$$A_0- d_0 N-\alpha Y=0 \qquad (6)$$

By using eq.(5) and eq.(6) we define the following function,

$$F(Y)=\{(\beta+\beta_0 N_0-\beta_0 A_0/d_0)\ A_0/d_0 - (\alpha+\mu+d)\ \}$$

$$+ \{\beta_0.(\alpha A_0)/d_0^2 -(\alpha+d_0))/d_0\ (\beta+\beta_0 N_0-\beta_0.A_0/d_0)\}Y-\{\alpha(\alpha+d_0)/d_0^2 .\beta_0\} Y^2=0 \qquad (7)$$

It is noted from the eq.(7), we have

(i) $F(0)=\{(\beta+\beta_0 N_0-(\beta_0 A_0)/d_0)\ A_0/d_0 -(\alpha+\mu+d)\}>0$

for. or $R_0=(\beta+\beta_0.N_0)\ A_0)/(\alpha+\mu+d).d_0)+(\beta_0 A_0^2)/d_0)>1$.

(ii) $F(A_0/\alpha+d_0)=-(\alpha+\mu+d)<0$

Hence, by the Intermediate value theorem the equation has at least one root in the interval. To show the uniqueness of root in the interval, we prove that . By differentiate eq. w.r.t. , we get

$F'(Y)=\{\beta\_0.(\alpha A_0)/d_0^2 -(\alpha+d_0)/d_0 (\beta+\beta_0 N_0-\beta_0.A_0/d_0)\}-\{\alpha(\alpha+d_0)/d_0^2.\beta_0 \}2Y$

(8)

Then, by using eq.(7) again, we have

$YF'(Y)=-\{(\beta+\beta_0 N_0-\beta_0 A_0/d_0) A_0/d_0 - (\mu+\alpha+d)\}-\{\alpha(\alpha+d_0)/d_0^2.\beta_0 \} Y_2$
(9)

Which is negative for $R_0>1$. Thus $F(Y)=0$ have a unique root in the interval $0<Y<A_0/\alpha+d_0$. Now by knowing the value of, the value of can be uniquely determined from eq.(6). Hence $E_1 (Y^*,N^*)$ exists if $R_0>1$.

Remark: From eq. (5) and eq. (6), it is easy to note that $dY/d\mu_0$ 0 and $dY/d\beta_0$ and $dY/d\beta_0 0$. This implies that , as μ_0 or β_0 or increases, infected population density decreases.

Stability Analysis: In this section, we study the stability behavior of equilibrium points. The local stability behavior of the equilibrium point $E_0 (0,A_0/d_0)$ can be investigated by determining the sign of the Eigen value of Jacobian matrix and the local stability behavior of can be investigated by considering suitable positive definite Lyapunov function. These results are given in the following theorems.

Theorem 4.1: The equilibrium point $E_0 (0,A_0/d_0)$ is unstable if and the equilibrium point $E_1 (Y^*,N^*)$ i is locally asymptotically stable provided the following inequality is satisfied.

$\alpha(\beta_0 (N^*-Y^*))^2<[4d_0 \{\beta-\beta_0 (N^*-N_0)\}^2$ (10)

Where $N^*-Y^*=(A_0-(\alpha+d_0)Y^{\wedge *})/d_0$

Proof: See Appendix A.

Theorem 4.2: The equilibrium point $E_1 (Y^*,N^*)$ is globally asymptotically stable in the region provided the following inequality is satisfied.

$\alpha(\beta_0.A_0/d_0)^2<4d_0 \{\beta-\beta_0 (N^*-N_0)\}^2$ (11)

Proof: See Appendix B.

Remark: It is noted that the inequalities (10) and (11) are automatically satisfied if $\beta_0=0$.

Numerical simulation and discussion. Here we discuss the existence and stability of the nontrivial equilibrium point E^* by considering the values of parameter from Table 1 and using the software MAPLE.

Table.1

Parameter	Value of parameter and references
A	500/ day [11]
	$1.98e^{-02}$ / day
	10000 persons per unit area
D	0.03/ day
	0.06/ day
	$1.2e^{-5}$ /person
	0.03/ day
	$1.98e^{-10}$ / person

For these values of parameters the nontrivial equilibrium point corresponding to eq. (5) and eq. (6) is obtained as

$$N^*=12008.81\cong 12009 , \quad Y^*=1665.96\cong 1666.$$

It may be noted that for the parameter values defined in Table.1, the condition $R_0>1$, and local and global stability condition are satisfied.

For the above values of parameters the Jacobian Matrix at (1666, 12009) is

$$M^*=\begin{bmatrix} -0.0193 & 0.0159 \\ -0.06 & -0.0498 \end{bmatrix}$$

The Eigen values of the Jacobian Matrix corresponding to the equilibrium point $E_1 (Y^*,N^*)$ are:

-0.0346 + 0269i , -0.0346 - 0269i

Both Eigen values are complex number having negative real parts. Thus, the equilibria $E_1 (Y^*,N^*)$ is asymptotically stable.

(i) The numerical simulation of model system (4) are also conducted and the results are shown in figures [1-6] from which the following results are concluded.

(ii) Fig.1 shows the global stability of the system.

(iii) Fig.2 shows increases as N_0 increases.

(iv) Fig.3 it is noted increases as *A* increases.

(v) Fig.4 it is seen increases as β increases.

(vi) Fig.5 it is noted decreases as μ_0 increases.

Fig.6 shows decreases as β_0 increases.

Fig.1. Stability diagram for infective population density (Y) and total population density (N)

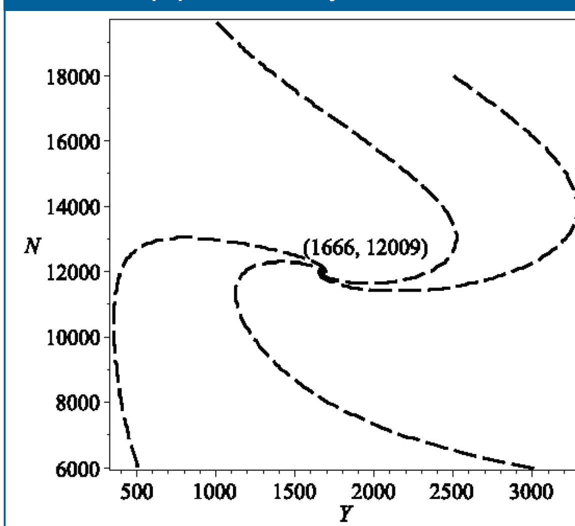


Fig.2: Infected population density (Y) Vs. time (t) in days for various values of non-emigrating population density (N_0).

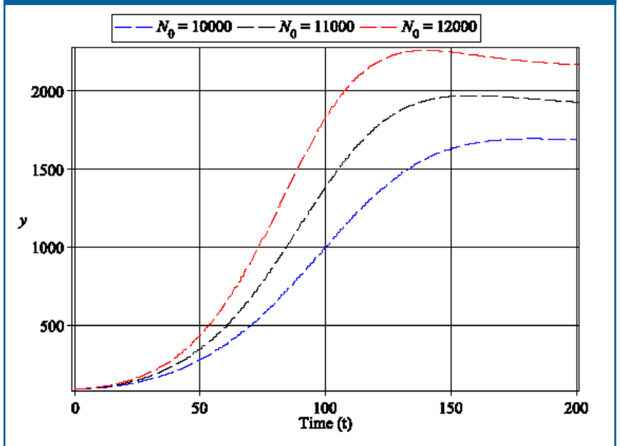


Fig.3: Infected population density (Y) Vs. time (t) in days for various values of constant immigration.

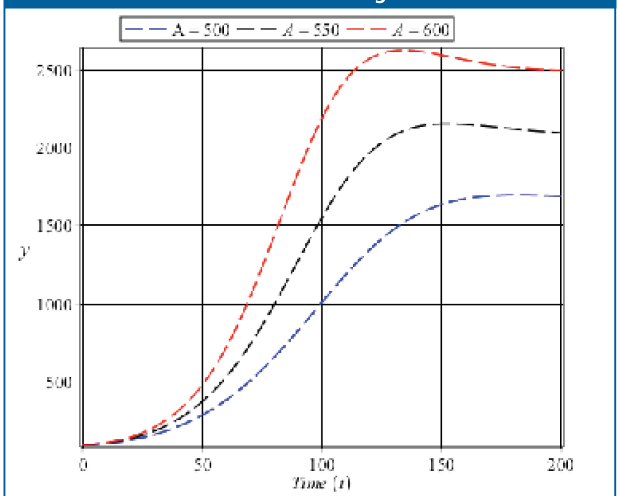


Fig.4: Infected population density (Y) Vs. time (t) in days for various values of constant contact rate (β).

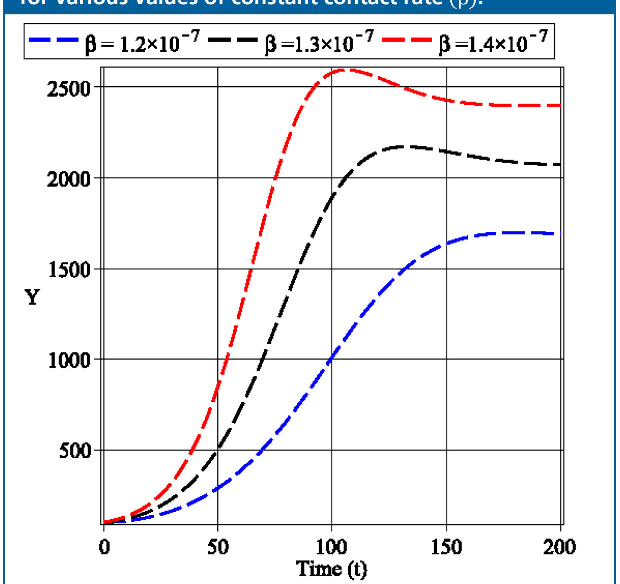


Fig.5: Infected population density (Y) Vs. time (t) in days for various values of variable emigration rate (μ_0).

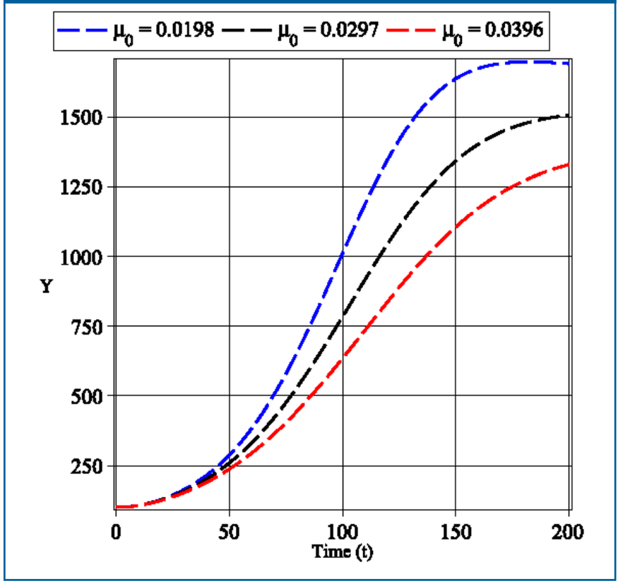
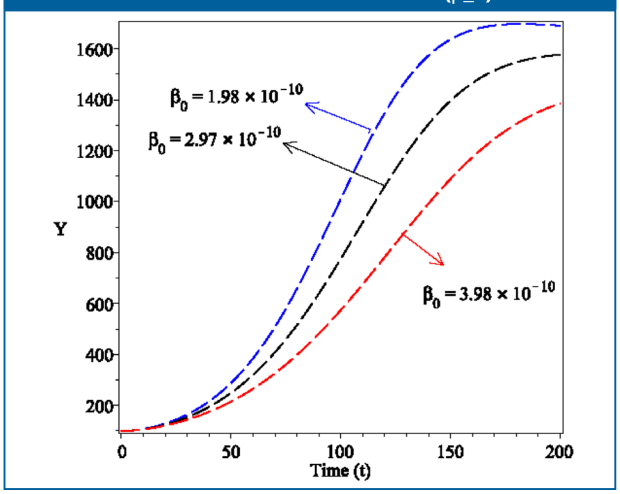


Fig.6: Infected population density (Y) Vs. time (t) in days for various values of variable contact rate (β_0).



Conclusions

In this paper, an SIS epidemic non-linear model have been proposed and analyzed to study the effect of density dependent emigration and non-emigrating population of habitat on spread of infectious diseases. In the modeling process, the two variables have been considered namely, the susceptible population density and the infective population density. The rate of contact between susceptible and infective has been assumed to be emigration dependent. The model has been analyzed by using the stability theory of differential equations and simulations. In the equilibrium analysis we found two non-negative equilibrium, one of them is disease free and the other is endemic equilibrium. In stability analysis we investigated the behavior of equilibrium points. The model analysis has shown that if the non-emigrating population density increases, the infective population density increases. Further as emigration increases, not only the contact rate decreases but the spread of infectious disease decreases.

The various results from simulation are summarized in the following.

- As the non-emigrating population density increases, the infective population density also increases i.e. the spread of infectious disease increases.
- As the constant immigrated population density increases, the infective population density also increases i.e. the spread of infectious disease increases.
- As the constant contact rate increases, infective population density increases.
- As the variable emigration rate increases, infective population density decreases
- As the variable contact rate that increases, the increases then infective population density decreases i.e. the spread of disease decreases.

Acknowledgment: The second Author (Jitendra Singh) is thankful to P.P.N (PG) College, Kanpur India and Innovative internet University Kanpur India for the help and support. I am also thankful to Prof. J.B Shukla, President of Indian Academy of Mathematical Modelling and Simulation for valuable support and suggestions to write this paper.

Appendix A. Proof of theorem 4.1.: The local stability behavior of each two equilibrium points and is studied by computing Jacobian Matrix at equilibrium points.

Let $f_1=(\beta-\beta_0 (N-N_0))(N-Y)Y-(d+\alpha+\mu)Y$

$f_2=A_0- d_0 N-\alpha Y$

Thus Jacobian matrix at E_0

$J(E_0)=[(\beta+\beta_0 N_0-\beta_0 A_0/d_0) A_0/d_0 - (\mu+\alpha+d) \ 0-\alpha]$
$-\alpha$

Since $R_0>1$, one of Eigen values $\lambda_1=(\beta+\beta_0 N_0-\beta_0 A_0/d_0) A_0/d_0 - (\mu+\alpha+d)>0$

Hence E_0 is Unstable.

Now we check the local stability of $E_1 (Y^*,N^*)$ of by using the Lyapunov's method, for this the following positive definite function is used.

$$V(y,n)=1/2y^2+K^1/2\,n^2 \quad A.1$$

By differentiating eq. (A.1), we getwe get

$$V^{\cdot}(y,n)=yy^{\cdot}+K\_1\ nn^{\cdot} \quad A.2$$

Now using linearization of the model system (4) about and by taking

$y=Y-Y^*$,$n=N-N^*$, we get

$$V^{\cdot}(y,n) = -Y^*.(\beta-\beta_0 (N\text{^}*-N_0)) .y^2$$

$$-(\beta_0 (N\text{^}*- Y^*) Y^*+(\beta-\beta_0 (N\text{^}*-N_0)).Y^*-\alpha K_1)ny-K_1 d_0 n^2 \quad A.3$$

By choosing such that $\Rightarrow K\_1=(Y\text{^}* (\beta -\beta_0 (N^*-N_0)))/\alpha$ then

$$V^{\cdot}(y,n)=-Y^* (\beta-\beta_0 (N^*-N_0)) y^2-(\beta_0 (N^*-Y^*) Y\text{^}*)ny-K_1 d_0 n^2 \quad A.4$$

Now $V^{\cdot}(y,n)<0$ if $(\beta_0 (N^*-Y^*) Y^*)\text{^}2<4K_1 d_0 Y^* (\beta-\beta_0 (N^*-N_0))$

i.e $\alpha(\beta_0 (N^*-Y^*))^2<4d_0 (\beta-\beta_0 (N\text{^}*-N_0))^2 \quad A.5$

$E_1 (Y^*,N^*)$ is locally stable providing inequality (A.5) is satisfied .

Appendix B. Proof of theorem 4.2.

To prove this theorem, we consider the following positive definite function

$$U=(Y-Y^*-Y^* \ln Y/Y^*)+(K_2 (N-N^*)^2)/2 \quad B.1$$

By differentiating B.1, we get,

$$U^{\cdot}=((Y-Y^*)/Y)\, Y^{\cdot}+K_2 (N-N^*)\, N^{\cdot} \quad B.2$$

Now after using model system (4), eq.(5) and eq.(6), we get

$$U^{\cdot}=((\beta-\beta_0 (N^*-N_0))-\alpha K_2-\beta_0 (N-Y))(N-N^*)(Y-Y^*)$$

$$- K_2 d_0 (N-N^*)^2- (\beta- \beta_0 (N^*-N_0)) (Y-Y^*)^2 \quad B.3$$

By choosing K_2 s.t $K_2=((\beta-\beta_0 (N\text{^}*-N_0))/\alpha$ then

$$U^{\cdot}=- (\beta- \beta_0 (N^*-N_0)) (Y-Y^*)^2-\beta_0 (N-Y)(N-N^*)(Y-Y^*) -K_2 d_0 (N-N^*)^2 \quad B.4$$

$U^{\cdot}<0$ iff $(\beta_0 (N-Y)^2<4K_2 d_0 (\beta-\beta_0 (N^*-N_0))$

$$\alpha(\beta_0 N_{max})^2<4d_0 (\beta-\beta_0 (N^*-N0))^2 \quad B.5$$

$$\alpha(\beta_0.A_0/d_0)^2<4d_0 \{\beta-\beta_0 (N^*-N_0)\}2 \quad B.6$$

$E_1 (Y^*,N^*)$ is globally asymptotically stable Providing B.6 satisfied.

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