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REPRESENTATION OF MEASURABLE POSITIVE DEFINITE GENERALIZED TOEPLITZ KERNELS IN $\mathbf{R}$

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We prove that every measurable positive definite generalized Toeplitz Kernel, defined in an (finite or infinite) interval (-a,a), is the sum of a positive definite generalized Toeplitz kernel given by continuous functions and a positive definite generalized Toeplitz kernel which vanishes almost everywhere. The proof is based on the theory of local semi-groups of contractions developed in former works. In the case of ordinary Topeplitz kernels this result gives theorems of F. Riesz, M. Krein and M. Crum and a special case of a theorem of Z. Sasvári.

INTRODUCTION

Let a be such that $0 < a \leq +\infty$ and let $I = (-a, a)$. A kernel on I is a function $K : I \times I \to \mathbf{C}$. K is said to be positive definite if for any positive integer n and any $x_1, ..., x_n$ in I, $\lambda_1, ..., \lambda_n$ in $\mathbf{C}$ we have

$$\sum_{i,j=1}^{n} K(x_i, x_j)\lambda_i\overline{\lambda_j} \geq 0$$

K is said to be a Toeplitz kernel if there exists a function $k : I - I \to \mathbf{C}$ such that $K(x, y) = k(x - y)$ for all x , y in I.

The Bochner theorem says that if $a = +\infty$ and $K(x, y) = k(x-y)$ is a continuous positive definite Toeplitz kenel on $I = \mathbf{R}$ then $K(x, y) = \hat{\mu}(x-y)$ where μ is a positive finite measure in $\mathbf{R}$, and the M. Krein extension theorem says that this is also true for $a < +\infty$.

F. Riesz [16] extended Bochner's theorem, by proving that every measurable positive definite Toeplitz kernel $K(x, y) = k(x - y)$ on $\mathbf{R}$ is equal almost everywhere to the Fourier transform of a positive finite Borel measure on $\mathbf{R}$.

That is, if $K : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ is a measurable positive definite Toeplitz kernel then $K = K^c + K^o$, where K^c and K^o are Toeplitz kernels, K^c is continuous and $K^o = 0$ almost everywhere.

M. Crum [10] proved that also the kernel K^o is positive definite.

In his book [15] Zoltán Sasvári makes the following comments: According to a remark of M. G. Krein [14], already Artjomenko, who lost his life in the second world war, knew that the kernel K^o is also positive definite, but he never published his proof. In 1943 Krein [13] announced an analogous result for positive definite kernels defined on the interval $I = (-a, a)$. Sasvári also proves (page 101) that if $0 < a < +\infty$ and $I = (-a, a)$ then every positive definite Toeplitz kernel K on I can be extended to a positive definite Toeplitz kernel F on $\mathbf{R}$. If K is measurable (continuous) on I then F is measurable (continuous) on $\mathbf{R}$ and (page 81) he gives a generalization of Crum's result for locally compact abelian groups.

Let $I = (-a, a), I_1 = I \bigcap [0, +\infty) = [0, a), I_2 = I \bigcap (-\infty, 0) = (-a, 0)$. A generalized Toeplitz kernel on I is a kernel $K : I \times I \to \mathbf{C}$ such that there are four functions $k_{\alpha\beta} : I_\alpha - I_\beta \to \mathbf{C}$ such that

$$K(x, y) = k_{\alpha\beta}(x - y) \text{ for all } (x, y) \in I_\alpha \times I_\beta$$

(cf [1], [2] , [7], [8], [9]). We will not suppose that the $k_{\alpha,\beta}$ functions are continuous.

The main result of this paper is the following: If K is a measurable positive definite generalized Toeplitz kernel on the interval I then $K = K^c + K^o$, where K^c and K^o are generalized Toeplitz kernels on $I = (-a, a)$, K^c is given by four continuous functions and K^o vanishes almost everywhere. This is a generalization of Crum's result [10] and a partial generalization of a result of Sasvári [15, page 101] to generalized Toeplitz kernels. The proof is based on the theory of local semigroups of contractions and isometries developed in [4], see also [6], [5] and [12].

PRELIMINARIES

REMARK: The theory of positive definite generalized Toeplitz kernels is closely related to the theory of bounded Hankel forms in weighted H^2 spaces and to the theorems of Nehari and Helson-Szegö. Therefore the results of this paper provide applications to Hankel forms which will be discussed elsewhere.

Let $0 < a \leq +\infty$ and let $I = [0, a)$.

A local semigroup of isometries (L.S.I.) on the Hilbert space $(H, \langle\ ,\ \rangle)$ is a family $(S_r, H_r)_{r \in [0,a)}$ such that:

(i) H_r is a closed subspace of H, $S_r : H_r \to H$ is an isometric operator, $H_t \subset H_r$ for $0 \leq r < t < a$ and $H_0 = H$, $S_0 = I_H$.

(ii) If $r, t \in [0, a)$ are such that $r + t < a$ then $S_t H_{r+t} \subset H_r$ and $S_{r+t} h = S_r S_t h$ for all $h \in H_{r+t}$.

(iii) $\bigcup_{r \in (x,a)} H_r$ is dense in H_x for all $x \in [0, a)$.

(iv) If $r \in [0,a)$ and $f \in H_r$ then the function $t \to S_t h$ is continuous on [0,r].

A local semigroup of isometries can be associated in a natural way to a positive definite generalized Toeplitz kernels given by continuous functions. We shall use the following result (for details see [4])

THEOREM A [4] *Let $(S_r, H_r)_{r\in[0,a)}$ be a local semigroup of isometries on the Hilbert space H. Then there exists a Hilbert space F, containing H as a closed subspace and a strongly continuous group of unitary operators $(U_t)_{-\infty<t<+\infty}$ on F such that $S_r = U_r |_{H_r}$ for all $r \in [0,a)$.*

THE MAIN RESULT

We shall use Theorem A to prove the following:

THEOREM 1 *Let $I = (-a,a)$ where $0 < a \leq +\infty$ and let K be a measurable positive definite generalized Toeplitz kernel on I.*

Then

$$K = K^c + K^o$$

where K^c and K^o are positive definite generalized Toeplitz kernels on I, K^c is given by four continuous functions and K^o vanishes almost everywhere.

The idea of the proof is the following: We are going to construct two Hilbert spaces $H_1(K)$ and $H_2(K)$. In the Hilbert space $H_2(K)$ we define a local semigroup of isometries and extending this semigroup to an unitary group we will obtain the kernel K^c, and then with geometrical arguments on $H_1(K)$, we will show that the kernel K^o vanishes almost everywhere.

In the sequel $I = (-a,a)$ and K is a measurable positive definite generalized Toeplitz kernel on I.

The proof will be done in several steps.

Construction of the Hilbert space $H_1(K)$

Let $E_1(K)$ be the set of the functions $p : I \to \mathbf{C}$ such that

$$p(x) = \sum_{i=1}^{n} p_i K(x, x_i)$$

where $n \in \mathbf{N}$, $p_1, ..., p_n \in \mathbf{C}$, $x_1, ..., x_n \in I$.

If p and q are elements of $E_1(K)$ and

$$p(x) = \sum_{i=1}^{n} p_i K(x, x_i) \qquad q(x) = \sum_{j=1}^{m} q_j K(x, y_j)$$

we define

$$\langle p, q \rangle_1 = \sum_{j=1}^{m} \sum_{i=1}^{n} p_i \overline{q_j} K(y_j, x_i)$$

It is clear that $\langle\ ,\ \rangle_1$ is a non-negative sesquilinear form on $E_1(K)$.

For $y \in I$ let $K_y(x) = K(x, y)$, then we have that for every $p \in E_1(K)$

$$p(y) = \langle p, K_y \rangle_1 \quad \text{for all } y \in I \tag{1}$$

therefore

$$| p(y) | = | \langle p, K_y \rangle_1 | \leq \| p \|_1 \| K_y \|_1 = \| p \|_1 K(0, 0) \tag{2}$$

where $\| \ \|_1$ denotes the norm associated with the product $\langle\ ,\ \rangle_1$.

Then $E_1(K)$ is a pre-Hilbert space and convergence in $E_1(K)$ implies uniform convergence.

$H_1(K)$ will denote the completion of $E_1(K)$. It is clear that such elements are measurable bounded functions and (1) and (2) are valid for $H_1(K)$ elements. Therefore K is a reproducing kernel for $H_1(K)$ in the sense of [3].

Construction of the Hilbert space $H_2(K)$

Let $E_2(K)$ be the set of the complex value continuous functions with compact support contained in I.

If $f, g \in E_2(K)$ we define

$$\langle f, g \rangle_2 = \int_{-a}^{a} \int_{-a}^{a} K(x, y) f(x) \overline{g(y)} dx dy \tag{3}$$

PROPOSITION 1 *The sesquilinear form $\langle\ ,\ \rangle_2$ is non-negative.*

REMARK: This assertion is not immediate since K is not supposed to be continuous and the usual argument based on Riemann sums cannot be used here.

Proof:

If $p \in E_1(K)$ then the function $u \to \langle K_u, p \rangle_1$ with domain I is bounded and measurable, its value in each u is $\overline{p(u)}$ and it is bounded by $\| p \|_1 K(0, 0)$.

Let $h : \mathbf{I} \to \mathbf{C}$ be a continuous function of compact support. Then the antilinear functional from $H_1(K)$ to $\mathbf{C}$

$$p \longrightarrow \int_{-a}^{a} h(u)\langle K_u, p\rangle_1 du$$

is continuous.

Therefore there exists an element $A(h)$ in $H_1(K)$ such that

$$\langle A(h), p\rangle_1 = \int_{-a}^{a} h(u)\langle K_u, p\rangle_1 du$$

for all p in $H_1(K)$.

Moreover

$$\begin{aligned} A(h)(y) = \ \langle A(h), K_y\rangle_1 &= \int_{-a}^{a} h(u)\langle K_u, K_y\rangle_1 du \\ &= \int_{-a}^{a} h(u)K_u(y)du \\ &= \int_{-a}^{a} h(u)K(y,u)du \end{aligned}$$

Finally if $h \in E_1(K)$ then

$$\begin{aligned} 0 \leq \ \langle A(h), A(h)\rangle_1 &= \int_{-a}^{a} h(u)\langle K_u, A(h)\rangle_1 du \\ &= \int_{-a}^{a} h(u)\left(\int_{-a}^{a} \overline{h(v)K(u,v)}dv\right)du \end{aligned}$$

$H_2(K)$ will be the Hilbert space obtained by completing $E_2(K)$, after the natural quotient.

$\|\ \|_2$ will denote the norm associated with the product $\langle\ ,\ \rangle_2$.

The local semigroup of isometries in the space $H_2(K)$.

We will construct a local semigroup of isometries $(S_r, H_r)_{r\in[0,a)}$ on $H_2(K)$ in the following way:

Let E_r be the set of the functions in $E_2(K)$ with support contained in $(-a+r, 0) \cup (r, a)$. For f in E_r we define $(S_r f)(x)$ as $f(x+r)$ if x is in $(-a+r, 0) \cup (r, a)$ and 0 in the rest. It is clear that the operators S_r are isometries. Let H_r be the closure of E_r in $H_2(K)$. Then the operators S_r can be extended to isometric operators from H_r in $H_2(K)$, this extension will be denoted by S_r also. We have the following result:

PROPOSITION 2 *The family* $(S_r, H_r)_{r\in[0,a)}$ *is a local semigroup of isometries on the Hilbert space* $H_2(K)$.

Proof:

(i), (ii) and (iii) are clear.

(iv) follows from the continuity of the function

$$t \to \int_{-a}^{a}\int_{-a}^{a} K(x,y)f(x+t)\overline{f(y)}dxdy$$

for f in E_r, $0 \leq t \leq r$

<u>Relation between K and the associated local semigroup</u>

For $n \in \mathbf{N}$ let φ_n^1 and φ_n^2 be the functions defined by

$$\varphi_n^1(x) = \begin{cases} n & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{in another case} \end{cases}$$

$$\varphi_n^2(x) = \begin{cases} n & \text{if } -\frac{1}{n} < x < 0 \\ 0 & \text{in another case} \end{cases}$$

It is easy to check that φ_n^1 and φ_n^2 are $H_2(K)$ elements.

For $t \in (-a, 0)$ let $\varphi_{n,t}^1$ the function defined by

$$\varphi_{n,t}^1(x) = \varphi_n^1(x+t)$$

For $t \in (0, a)$ let $\varphi_{n,t}^2$ the function defined by

$$\varphi_{n,t}^2(x) = \varphi_n^2(x+t)$$

For $\alpha = 1, 2$

$$\varphi_{n,0}^\alpha = \varphi_n^\alpha$$

By a classical result in measure theory, see for example [11, page 216, Corollary 7], if $F : \mathbf{R}^2 \to \mathbf{C}$ is a bounded and measurable function then for $\alpha, \beta = 1, 2$

$$\lim_{n\to\infty} \int_{-a}^{a}\int_{-a}^{a} F(x,y)\varphi_{n,t}^\alpha(x)\varphi_{n,t}^\beta(y)dxdy = F(t,r) \tag{4}$$

at almost every point $(t, r) \in I_\alpha \times I_\beta$.

If we put $F(x,y) = K(x,y)$ we obtain, for $\alpha, \beta = 1, 2$

$$\lim_{n\to\infty} \int_{-a}^{a}\int_{-a}^{a} K(x,y)\varphi_{n,t}^\alpha(x)\varphi_{n,t}^\beta(y)dxdy = k_{\alpha\beta}(t-r) \tag{5}$$

at almost every point $(t, r) \in I_\alpha \times I_\beta$.

PROPOSITION 3 *For* $\alpha = 1, 2$ *and for all* $t \in I_\alpha \cup \{0\}$ *the sequence* $\{\varphi^\alpha_{n,t}\}_{n=1}^{+\infty}$ *is weakly convergent in* $H_2(K)$

Proof:

K is bounded (because it is positive definite). From the definition of the norm on $H_2(K)$ it follows that the sequence $\{\| \varphi^\alpha_{n,t} \|_2\}_{n=1}^{+\infty}$ is bounded. If $f \in E_2(K)$ then

$$\begin{aligned} \lim_{n\to\infty} \langle \varphi^\alpha_{n,t}, f\rangle_2 &= \lim_{n\to\infty} \int_{I_\alpha} \int_{-a}^{a} K(x,y)\varphi^\alpha_{n,t}(x)\overline{f(y)}dxdy \\ &= \int_{-a}^{a} K(t,y)\overline{f(y)}dy \end{aligned}$$

Since $E_2(K)$ is dense in $H_2(K)$ we have that $\lim_{n\to\infty}\langle \varphi^\alpha_{n,t}, f\rangle_2$ exists for all $f \in H_2(K)$

For $\alpha, \beta = 1, 2$ and $t \in I_\alpha \cup \{0\}$ let δ^α_t be the weak limit of the sequence $\varphi^\alpha_{n,t}$.

From the proof of the last proposition it follows that

PROPOSITION 4 *If* $f \in E_2(K)$ *and* $t \in I_\alpha \cup \{0\}$ *then* $\langle \delta^\alpha_t, f\rangle_2 = \int_{-a}^{a} K(t,y)\overline{f(y)}dy$

It is clear that we have

$$S_{-t}\delta^1_0 = \delta^1_t \;\; if\; t \in (-a, 0) \tag{6}$$

$$S_t\delta^2_t = \delta^2_0 \;\; if\; t \in [0, a) \tag{7}$$

From proposition 4 and the definition of δ^α_t we have that for $\alpha,\ \beta = 1, 2$

$$\langle \delta^\alpha_t, \delta^\beta_r\rangle_2 = k_{\alpha\beta}(t-r) \quad \text{at almost every point}(t,r) \in I_\alpha \times I_\beta \tag{8}$$

<u>Construction of the function K^c</u>

By theorem A there exists a Hilbert space F, which contains $H_2(K)$, and a strongly continuous group of unitary operators $(U_t)_{-\infty<t<+\infty}$ such that $S_t = U_t \mid_{H_t}$ for all $t \in [0, a)$.

From (6), (7) and $U_{-t} = U_t^{-1}$ it follows that

$$U_{-t}\delta^1_0 = \delta^1_t \;\; if\; t \in (-a, 0) \tag{9}$$

$$U_{-t}\delta^2_0 = \delta^2_t \;\; if\; t \in [0, a) \tag{10}$$

Using (8) we obtain

$$k_{\alpha\beta}(t-r) = \langle U_{-t}\delta^\alpha_0, U_{-r}\delta^\beta_0\rangle_2 = \langle U_r\delta^\alpha_0, U_t\delta^\beta_0\rangle_F \tag{11}$$

at almost every point $(t,r) \in I_\alpha \times I_\beta$.

It is easy to check that (see [4]) the generalized Toeplitz kernel on I, K^c given by the functions $k^c_{\alpha\beta}(t-r) = \langle U_r\delta^\alpha_0, U_t\delta^\beta_0\rangle_F$ is positive definite. It is clear that the functions $k^c_{\alpha\beta}$ are continuous. So, we have proved

$$K = K^c + K^o$$

where K^c and K^o are generalized Toeplitz kernels on I , K^c is positive definite, given by four continuous functions and K^o is null almost everywhere.

It remains only to proof that K^o is definite positive.

K^o is definite positive

From the proof of the proposition 1 it follows that the function

$$h \to A(h)$$

from $E_2(K)$ to $H_1(K)$ is linear and isometric. Therefore it can be extended to an isometric operator from $H_2(K)$ in $H_1(K)$, this extension will also be denote by A.

Since $K = K^c$ at almost every point we have that

$$A(h)(x) = \int_{-a}^{a} h(u)K^c(x,u)du$$

for all $x \in (-a,a)$.

Therefore if $t \in I_\alpha$ then

$$A(\delta^\alpha_t)(x) = K^c(x,t)$$

for all $y \in (-a,a)$.

$A(H_2(K))$ is a closed subspace of $H_1(K)$. So it is clear that the function K^c_t given by $K^c_t(x) = K^c(x,t)$ is an $H_1(K)$ element. Since $K = K^c + K^o$ we have that the function K^o_t given by $K^o_t(x) = K^o(x,t)$ is an $H_1(K)$ element also.

Let $x,y \in I = (-a,a)$, then if $x \in I_\alpha$ and $y \in I_\beta$

$$\langle K^c_x, K^c_y\rangle_1 = \langle A(\delta^\alpha_x), A(\delta^\beta_y)\rangle_1 = \langle \delta^\alpha_x, \delta^\beta_y\rangle_2 = K^c(x,y)$$

$$\langle K^c_x, K_y\rangle_1 = K^c(x,y)$$

Since $K_y = K^c_y + K^o_y$ it must be $\langle K^c_x, K^o_y\rangle_1 = 0$ and therefore

$$\langle K^o_x, K^o_y\rangle_1 = \langle K_x - K^c_x, K^o_y\rangle_1 = K^o(x,y)$$

Finally let $n \in \mathbf{N}$, $x_1, ..., x_n \in I$, $\lambda_1, ..., \lambda_n \in \mathbf{C}$

$$\sum_{i,j=1}^{n} K^o(x_i, x_j)\lambda_i\overline{\lambda_j} = \langle \sum_{i=1}^{n} \lambda_i K^o_{x_i}, \sum_{j=1}^{n} \lambda_j K^o_{x_j} \rangle_1 \geq 0$$

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